Oscillations near an Isosceles-Triangle Solution of the Problem of Three Bodies as the Finite Masses Become Unequal.

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§ 1. *Introduction.*

In an article entitled "Oscillations near one of the Isosceles-Triangle Solutions of the Three-Body Problem," which appeared in the July (1915) number of the *Proceedings of the London Mathematical Society*, the author of the present paper discussed the periodic oscillations of an infinitesimal body about a straight line drawn through the centre of gravity of two finite bodies of equal mass and perpendicular to the plane of their motion, which was assumed to be circular. In the problem now under consideration the finite bodies are assumed to be of unequal mass and the third body is assumed to be infinitesimal. The finite bodies are started so that they move in circles, and the infinitesimal body oscillates about the straight line through the centre of mass of the finite bodies and perpendicular to the plane of their motion, as in the former article. Initial conditions are determined so that the oscillations shall be periodic. The solutions for the motion of the infinitesimal body are expansible as power series in a certain parameter ε which represents half the difference in mass between the finite bodies. When $\varepsilon=0$ the solutions reduce to the simplest case of the isosceles-triangle solutions, viz., that in which the finite bodies are of equal mass and move in circles and the third body is infinitesimal.

As we shall have frequent occasion to refer to the former paper we shall refer to it as *Proc.*, followed by the number of the equations or the section.

§ 2. *The Differential Equations.*

Let m_1 and m_2 denote the two finite bodies of mass $m-\varepsilon$ and $m+\varepsilon$, respectively, and let μ denote the infinitesimal body. Let the unit of mass be so chosen that $m=1/2$, the linear unit so that the distance from m_1 to m_2 shall be unity, and the unit of time so that the Gaussian constant shall also be unity. Let the

origin of coordinates be taken at the centre of mass, the plane of motion of m_1 and m_2 as the $\xi\eta$-plane, and let the coordinates of m_1, m_2 and μ be denoted by ξ_1, η_1, ζ_1; ξ_2, η_2, ζ_2 and ξ, η, ζ, respectively. If the masses m_1 and m_2 are started from the points $1/2+\varepsilon$, 0, 0, and $-1/2+\varepsilon$, 0, 0, respectively, so that they will move in circles, then

$$\left.\begin{aligned} \xi_1 &= (\tfrac{1}{2}+\varepsilon)\cos(t-t_0), & \xi_2 &= -(\tfrac{1}{2}-\varepsilon)\cos(t-t_0), \\ \eta_1 &= (\tfrac{1}{2}+\varepsilon)\sin(t-t_0), & \eta_2 &= -(\tfrac{1}{2}-\varepsilon)\sin(t-t_0), \end{aligned}\right\} \tag{1}$$

and the differential equations of motion for the infinitesimal body are

$$\left.\begin{aligned} \xi'' &= -\frac{\xi}{2}\left[\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right]+\frac{1}{2}\left[\frac{\xi_1}{\rho_1^3}+\frac{\xi_2}{\rho_2^3}\right]+\varepsilon\xi\left[\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right]-\varepsilon\left[\frac{\xi_1}{\rho_1^3}-\frac{\xi_2}{\rho_2^3}\right], \\ \eta'' &= -\frac{\eta}{2}\left[\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right]+\frac{1}{2}\left[\frac{\eta_1}{\rho_1^3}+\frac{\eta_2}{\rho_2^3}\right]+\varepsilon\eta\left[\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right]-\varepsilon\left[\frac{\eta_1}{\rho_1^3}-\frac{\eta_2}{\rho_2^3}\right], \\ \zeta'' &= -\frac{\zeta}{2}\left[\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right]+\varepsilon\zeta\left[\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right], \\ \rho_1 &= [(\xi-\xi_1)^2+(\eta-\eta_1)^2+\zeta^2]^{\frac{1}{2}}, \\ \rho_2 &= [(\xi-\xi_2)^2+(\eta-\eta_2)^2+\zeta^2]^{\frac{1}{2}}. \end{aligned}\right\} \tag{2}$$

If we refer the motion of the system to a set of rectangular coordinates rotating about the ζ-axis with the uniform velocity unity, that is, if we transform the coordinates of μ by the substitutions

$$\left.\begin{aligned} \xi &= x\cos(t-t_0)-y\sin(t-t_0), \\ \eta &= x\sin(t-t_0)+y\cos(t-t_0), \\ \zeta &= z, \end{aligned}\right\} \tag{3}$$

then, after (1) and (3) are substituted in (2), the differential equations become

$$\left.\begin{aligned} x''-2y'-x &= -\frac{x}{2}\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]+\left[\frac{1}{4}+\varepsilon x-\varepsilon^2\right]\left[\frac{1}{r_1^3}-\frac{1}{r_2^3}\right], \\ y''+2x'-y &= -\frac{y}{2}\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]+\varepsilon y\left[\frac{1}{r_1^3}-\frac{1}{r_2^3}\right], \\ z'' &= -\frac{z}{2}\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]+\varepsilon z\left[\frac{1}{r_1^3}-\frac{1}{r_2^3}\right], \\ r_1 &= [(\tfrac{1}{2}+\varepsilon-x)^2+y^2+z^2]^{\frac{1}{2}}, \quad r_2 = [(\tfrac{1}{2}-\varepsilon+x)^2+y^2+z^2]^{\frac{1}{2}}. \end{aligned}\right\} \tag{4}$$

These equations admit the Jacobian integral

$$(x')^2+(y')^2+(z')^2=x^2+y^2+\frac{1-2\varepsilon}{r_1}+\frac{1+2\varepsilon}{r_2}+\text{const.} \tag{5}$$

When $\varepsilon=0$ and the infinitesimal is projected along the z-axis, then $x\equiv y\equiv 0$ and the differential equations (4) reduce to

$$z''=-\frac{8z}{(1+4z^2)^{\frac{3}{2}}}.$$

The periodic solution of this equation is, *Proc.* (4),

$$z=\psi=a\sin(\tau-\tau_0)+\frac{3}{16}a^3[\sin 3(\tau-\tau_0)-\sin(\tau-\tau_0)]+\ldots,$$

$$\tau-\tau_0=\frac{(t-t_0)}{\sqrt{\frac{1}{8}(1+\delta)}},\quad \delta=\frac{9}{2}a^2-\frac{141}{32}a^4+\frac{35}{2}a^6+\ldots.$$

The constant a is a variable parameter and $a/\sqrt{\{\frac{1}{8}(1+\delta)\}}$ denotes the initial projection of μ from the origin when $\varepsilon=0$. The numerical values of t_0 and τ_0 can both be taken to be zero without loss of generality. As series similar in form to ψ occur frequently in the sequel, we shall call them *triply odd series* inasmuch as they contain only odd powers of a and odd functions of odd multiples of τ.

§ 3. *The Equations of Variation.*

We wish to show the existence of periodic solutions of (4) which are expansible as power series in ε and which for $\varepsilon=0$ reduce to $x\equiv y\equiv 0$, $z=\psi$. Let us substitute in (4)

$$(t-t_0)=(\tau-\tau_0)\sqrt{\tfrac{1}{8}(1+\delta)},\quad z=\psi+w, \tag{6}$$

where w vanishes with ε. If we denote derivatives with respect to τ by dots, and expand the right members as power series in ε, x, y and w, then the differential equations (4) become

$$\left.\begin{aligned}
&\ddot{x}-2\sqrt{\tfrac{1}{8}(1+\delta)}\,\dot{y}-\tfrac{1}{8}(1+\delta)x=x\sum_{i=0}^{\infty}X_i^{(0)}+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}X_i^{(j)}\varepsilon^j,\\
&\ddot{y}+2\sqrt{\tfrac{1}{8}(1+\delta)}\,\dot{x}-\tfrac{1}{8}(1+\delta)y=y\sum_{i=0}^{\infty}Y_i^{(0)}+y\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}Y_i^{(j)}\varepsilon^j,\\
&\ddot{w}+W_0w=\sum_{i=2}^{\infty}W_i^{(0)}+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}W_i^{(j)}\varepsilon^j.
\end{aligned}\right\} \tag{7}$$

Before characterizing the undefined terms in (7), we shall define a *triply even series*, as such series occur frequently in the sequel. A triply even series is a power series in a^2 with sums of cosines of even multiples of τ in the coefficients. The highest multiple of τ in the coefficient of a^{2k} is $2k$. Such a series is called triply even since it is even in a and only even functions of even multiples of τ enter the coefficients.

The $X_i^{(0)}$, $Y_i^{(0)}$ and $W_i^{(0)}$ are homogeneous polynomials of degree i in x, y and w, but only even powers of x, and y enter while w may enter to any power. The coefficients in $X_i^{(0)}$ and $Y_i^{(0)}$ are triply even or triply odd series, according as i is even or odd, respectively. The coefficients in $W_i^{(0)}$ are triply odd or triply even series, according as i is even or odd, respectively. As far as the computations have been carried out we have, *Proc.* (5),

$$
\begin{aligned}
W_0 &= 1-a^2\left(\frac{9}{2}-9\cos 2\tau\right)+a^4\left(\frac{687}{32}-48\cos 2\tau+\frac{177}{8}\cos 4\tau\right)+\ldots\ldots,\\
X_0^{(0)} &= 2-3\,a^2(1-4\cos 2\tau)+\ldots\ldots,\\
Y_0^{(0)} &= -1-\frac{3}{2}a^2(1+2\cos 2\tau)+\ldots\ldots,\\
X_1^{(0)} &= -48(1+\delta)\,\psi\, w,\quad Y_1^{(0)}=12(1+\delta)\,\psi\, w,\\
X_2^{(0)} &= x^2[16-24\,a^2(7-10\cos 2\tau)+\ldots\ldots]-y^2[24-36\,a^2(2-5\cos 2\tau)+\ldots\ldots]\\
&\quad -w^2[24-108\,a^2(4-5\cos 2\tau)+\ldots\ldots],\\
Y_2^{(0)} &= -x^2[24-36\,a^2(2-5\cos 2\tau)+\ldots\ldots]+y^2[6-3\,a^2(1-10\cos 2\tau)+\ldots\ldots]\\
&\quad +w^2[6-9a^2(7-10\cos 2\tau)+\ldots\ldots],\\
W_2^{(0)} &= -x^2[24\,a\sin\tau+\ldots\ldots]+y^2[6\,a\sin\tau+\ldots\ldots]+w^2[18\,a\sin\tau+\ldots\ldots].
\end{aligned}
$$

The $X_i^{(j)}$, $Y_i^{(j)}$ and $W_i^{(j)}$ for $j>0$ are likewise polynomials of degree i in x, y and w, but y enters only to even degrees. If j is odd, the $X_i^{(j)}$ are even in x, and $Y_i^{(j)}$ and $W_i^{(j)}$ are odd in x. If j is even, the $X_i^{(j)}$ are odd in x, and $Y_i^{(j)}$ and $W_i^{(j)}$ are even in x. The coefficients in $X_i^{(j)}$ and $Y_i^{(j)}$ are triply even or triply odd series, according as w enters to even or odd degrees, respectively, but in $W_i^{(j)}$ the coefficients are triply odd or triply even series, according as w enters to even or odd degrees, respectively. It is observed that $X_0^{(j)}=0$ if j is even, and $Y_0^{(j)}=W_0^{(j)}=0$ if j is odd.

The expansions in the right members in (7) are valid only in a certain region of convergence. It is seen from the values of r_1 and r_2 in equations (4) that x, y and z must satisfy the inequalities

$$
\begin{aligned}
-\tfrac{1}{4} &< +\varepsilon+\varepsilon^2-x-2\varepsilon x+x^2+y^2+z^2<\tfrac{1}{4},\\
-\tfrac{1}{4} &< -\varepsilon+\varepsilon^2+x-2\varepsilon x+x^2+y^2+z^2<\tfrac{1}{4}.
\end{aligned}
$$

The region of convergence is obtained by replacing the inequality signs with signs of equality. When referred to the fixed $\xi\eta$-axes the region is found to be the spindle formed by rotating the common portion of two intersecting circles about their common chord. These circles have their centres at $1/2+\varepsilon$, 0, 0 and $-1/2+\varepsilon$, 0, 0 and have radii $1/\sqrt{2}$.

The equations of variation are obtained by putting $\varepsilon=0$ in (7) and taking only the linear terms in x, y and w in the right members. They are

$$\left.\begin{array}{l} \ddot{x}-2\sqrt{\{\frac{1}{8}(1+\delta)\}}\,\dot{y}-[\frac{1}{8}(1+\delta)+X_0^{(0)}]x=0, \\ \ddot{y}+2\sqrt{\{\frac{1}{8}(1+\delta)\}}\,\dot{x}-[\frac{1}{8}(1+\delta)+Y_0^{(0)}]y=0, \\ \ddot{w}+W_0 w=0. \end{array}\right\} \quad (8)$$

These equations are the same as *Proc.* (6) if we put $m_0=\frac{1}{2}$ in the latter. The solutions of the first two equations are obtained from *Proc.* (8) by putting $m_0=\frac{1}{2}$. They are

$$\left.\begin{array}{l} x = A_1 e^{\sqrt{-1}\beta_1\tau}\,u_1+A_2 e^{-\sqrt{-1}\beta_1\tau}\,u_2+A_3 e^{\beta_2\tau}\,u_3+A_4 e^{-\beta_2\tau}\,u_4, \\ y = \sqrt{-1}\,[A_1 e^{\sqrt{-1}\beta_1\tau}\,v_1-A_2 e^{-\sqrt{-1}\beta_1\tau}v_2]+A_3 e^{\beta_2\tau}\,v_3-A_4 e^{-\beta_2\tau}\,v_4, \end{array}\right\} \quad (9)$$

where A_i $(i=1,2,3,4)$ are the constants of integration, u_i and v_i $(i=1,2,3,4)$ are periodic functions of τ having the period 2π; and

$$\beta_j = \sum_{k=0}^{\infty} \beta_j^{(2k)} a^{2k} \quad (j=1,2)$$

are power series in a^2 with real constant coefficients which are determined by the conditions that u_i and v_i shall be periodic. The values of $-(\beta_1^{(0)})^2$ and $(\beta_2^{(0)})^2$ are the roots of the quadratic

$$64\gamma^2-48\gamma-119=0,$$

and, therefore,

$$\beta_1^{(0)}=[\sqrt{2}-\tfrac{3}{8}]^{\frac{1}{2}}, \quad \beta_2^{(0)}=[\sqrt{2}+\tfrac{3}{8}]^{\frac{1}{2}}.$$

The u_i and v_i are power series in a^2 with sums of sines and cosines of even multiples of τ in the coefficients, the highest multiple of τ in the coefficient of a^{2k} being $2k$. In u_i and v_i $(i=1,2)$ the coefficients of the sines are purely imaginary and the coefficients of the cosines are real. In u_i and v_i $(i=3,4)$ all the coefficients are real. Further,

$$\begin{array}{ll} u_1(\sqrt{-1}) = u_2(-\sqrt{-1}), & u_3(\tau) = u_4(-\tau), \\ v_1(\sqrt{-1}) = v_2(-\sqrt{-1}), & v_3(\tau) = v_4(-\tau), \\ \multicolumn{2}{c}{u_i(0) = 1 \;\; (i=1,2,3,4).} \end{array}$$

The determinant of the solutions (9) and their first derivatives is a constant,* and at $\tau=0$ its value is

$$\Delta = 10\sqrt{-119} + \text{terms in } a^2. \quad (10)$$

This determinant is different from zero for $a=0$ and will remain different from zero for a^2 sufficiently small. Hence, the solutions (9) constitute a fundamental set.

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* Moulton's "Periodic Orbits," § 18.

The general solution of the last equation of (9) is the same as *Proc.* (7), viz.,

$$\left.\begin{aligned} w &= A_5\phi + A_6[\chi + A\tau\phi], \\ \phi &= \cos\tau + \frac{9}{16}a^2[\cos 3\tau - \cos\tau] + \ldots, \\ \chi &= \sin\tau + \frac{9}{16}a^2[\sin 3\tau + 5\sin\tau] + \ldots, \\ A &= -\frac{9}{2}a^2 + \frac{141}{16}a^4 + \ldots, \end{aligned}\right\} \quad (11)$$

where A_5 and A_6 are the constants of integration.

§ 4. *Existence of Symmetrical Periodic Orbits.*

Let us choose as initial conditions of (7)

$$x(0) = \alpha_1,\ \dot{x}(0) = 0,\ y(0) = 0,\ \dot{y}(0) = \alpha_2,\ w(0) = 0,\ \dot{w}(0) = \alpha_3. \quad (12)$$

With these initial conditions it can be shown from the differential equations (7) that x is even in τ and y and w are odd in τ. Hence, sufficient conditions that the solutions shall be periodic with the period 2π are

$$\dot{x} = y = w = 0 \text{ at } \tau = \pi. \quad (13)$$

When the conditions (12) are imposed on the solutions (9) and (11), we find that $A_1 = A_2$, $A_3 = A_4$, $A_5 = 0$, and that A_1, A_3 and A_6 are linear functions of $\alpha_i (i=1,2,3)$, vanishing with α_i. Now let us integrate equations (7) as power series in ε and α_i, or, which is more convenient, in ε, A_1, A_3 and A_6. In so far as the terms which are linear in A_1, A_3 and A_6 are concerned, we obtain

$$\left.\begin{aligned} x_{10} &= A_1[e^{\sqrt{-1}\beta_1\tau}u_1 + e^{-\sqrt{-1}\beta_1\tau}u_2] + A_3[e^{\beta_2\tau}u_3 + e^{-\beta_2\tau}u_4], \\ y_{10} &= \sqrt{-1}A_1[e^{\sqrt{-1}\beta_1\tau}v_1 - e^{-\sqrt{-1}\beta_1\tau}v_2] + A_3[e^{\beta_2\tau}v_3 - e^{-\beta_2\tau}v_4], \\ w_{10} &= A_6[\chi + A\tau\phi]. \end{aligned}\right\} \quad (14)$$

When the periodicity conditions (13) are imposed on these solutions, we obtain

$$\left.\begin{aligned} 0 &= A_1[\sqrt{-1}\beta_1 + \dot{u}_1(0)][e^{\sqrt{-1}\beta_1\pi} - e^{-\sqrt{-1}\beta_1\pi}] + A_3[\beta_2 + \dot{u}_3(0)][e^{\beta_2\pi} - e^{-\beta_2\pi}] \\ &\qquad + \text{terms in } \varepsilon \text{ and higher degree terms in } A_1, A_3, A_6 \text{ and } \varepsilon, \\ 0 &= \sqrt{-1}v_1(0)A_1[e^{\sqrt{-1}\beta_1\pi} - e^{-\sqrt{-1}\beta_1\pi}] + v_3(0)A_3[e^{\beta_2\pi} - e^{-\beta_2\pi}] \\ &\qquad + \text{higher degree terms in } A_1, A_3, A_6 \text{ and } \varepsilon, \\ 0 &= A_6A\pi + \text{higher degree terms in } A_1, A_3, A_6 \text{ and } \varepsilon. \end{aligned}\right\} \quad (15)$$

The determinant of the linear terms in A_1, A_3 and A_6 is

$$\begin{aligned} D = {} & [e^{\sqrt{-1}\beta_1\pi} - e^{-\sqrt{-1}\beta_1\pi}][e^{\beta_2\pi} - e^{-\beta_2\pi}] \\ & \left\{v_3(0)[\sqrt{-1}\beta_1 + \dot{u}_1(0)] - \sqrt{-1}v_1(0)[\beta_2 + \dot{u}_3(0)]\right\}. \end{aligned} \quad (16)$$

The last factor has the form

$$\frac{5\sqrt{-119}}{7\sqrt{2}} + \text{terms in } a^2,$$

and is different from zero for a^2 sufficiently small. The second factor can not vanish as β_2 is real. Then the determinant D can vanish only when β_1 is an integer. When β_1 is not an integer, D is not zero; and hence equations (15) can be solved uniquely for A_1, A_3 and A_6 as power series in ε which vanish with ε and converge for $|\varepsilon|$ sufficiently small. Therefore symmetrical periodic solutions of (7) exist when β_1 is not an integer, and they have the form

$$x = \sum_{j=1}^{\infty} x_j \varepsilon^j, \quad y = \sum_{j=1}^{\infty} y_j \varepsilon^j, \quad w = \sum_{j=1}^{\infty} w_j \varepsilon^j, \tag{17}$$

where each x_j, y_j and w_j is separately periodic with the period 2π in τ.

From the following considerations of the differential equations (7) we show that x and y are odd in ε, and w is even in ε. The right member of the first equation in (7) is odd in x and ε, considered together, and even in y. The right member of the second equation is even in x and ε, considered together, and also even in y. Let the solutions (17) be denoted by

$$x = x(\varepsilon), \quad y = y(\varepsilon), \quad w = w(\varepsilon).$$

If the signs of x, y and ε be changed, the differential equations remain unchanged, and therefore,

$$x = -x(-\varepsilon), \quad y = -y(-\varepsilon), \quad w = w(-\varepsilon)$$

are also solutions. Since the solutions as power series in ε are unique, then

$$x(\varepsilon) = -x(-\varepsilon), \quad y(\varepsilon) = -y(-\varepsilon), \quad w(\varepsilon) = w(-\varepsilon),$$

or, x and y are odd in ε and w is even in ε. Hence, the periodic solutions of (7) have the form

$$x = \sum_{j=0}^{\infty} x_{2j+1}\, e^{2j+1}, \quad y = \sum_{j=0}^{\infty} y_{2j+1}\, e^{2j+1}, \quad w = \sum_{j=1}^{\infty} w_{2j}\, \varepsilon^{2j}. \tag{18}$$

The same result would be obtained in the construction of the solutions if the forms (17) were used instead of (18).

§ 5. *Existence of Symmetrical Periodic Orbits when β_1 is an Integer.*

When β_1 is an integer, the determinant (16) vanishes and the terms in (15) of higher degree in ε, A_1, A_3 and A_6 must be considered in order to establish the existence of symmetrical periodic orbits. We take the same initial conditions (12), and as in the previous section we integrate (7) as power series in A_1, A_3, A_6 and ε. In addition to the linear terms already obtained in (14), we need the quadratic terms in A_1, A_3 and A_6, and the term in A_1^3 in so far as it enters x. If these terms are denoted by x_{11}, y_{11}, w_{11} and x_{30}

respectively, then all these terms except w_{11} can be obtained from *Proc.* § 6 by putting $m_0=1/2$, $\lambda=0$ in the solutions there denoted by the same notation. Thus,

$$\left.\begin{aligned}
x_{11} &= \frac{1}{\Delta}A_1A_6Q_1\tau[e^{\sqrt{-1}\beta_1\tau}u_1-e^{-\sqrt{-1}\beta_1\tau}u_2]\\
&\qquad\qquad+\frac{\sqrt{-1}}{\Delta}A_3A_6Q_2\tau[e^{\beta_2\tau}u_3-e^{-\beta_2\tau}u_4],\\
y_{11} &= \frac{\sqrt{-1}}{\Delta}A_1A_6Q_1\tau[e^{\sqrt{-1}\beta_1\tau}v_1+e^{-\sqrt{-1}\beta_1\tau}v_2]\\
&\qquad\qquad+\frac{\sqrt{-1}}{\Delta}A_3A_6Q_2\tau[e^{\beta_2\tau}v_3+e^{-\beta_2\tau}v_4],\\
x_{30} &= \frac{1}{\Delta}A_1^3Q_3\tau[e^{\sqrt{-1}\beta_1\tau}u_1-e^{-\sqrt{-1}\beta_1\tau}u_2],
\end{aligned}\right\}\qquad(19)$$

where Q_1, Q_2 and Q_3 are power series in a^2 with real constant coefficients.

The term w_{11} is the solution of the differential equation

$$\ddot{w}_{11}+W_0w_{11}=W_{11},\qquad(20)$$

where the right member is a linear function of x_{10}^2, y_{10}^2, w_{10}^2 multiplied by a triply odd series. The complementary function of (20) is the same as the solution of the last equation of (8), and on using the method of the variation of parameters we obtain

$$w_{11}=\tau[A_1^2\phi_1+A_3^2\phi_3]+A_6^2[\text{non-periodic terms}]+\text{periodic terms},$$

where ϕ_1 and ϕ_3 are power series similar to ϕ in (11).

When the periodicity conditions (13) are imposed on the solutions

$$\begin{aligned}
x&=x_{10}+x_{11}+x_{30}+\ldots.,\\
y&=y_{10}+y_{11}+\ldots.,\\
w&=w_{10}+w_{11}+\ldots.,
\end{aligned}$$

we obtain the equations

$$\left.\begin{aligned}
0 &= A_3[\beta_2+\dot{u}_3(0)]\left\{e^{\beta_2\pi}-e^{-\beta_2\pi}+\frac{\sqrt{-1}\pi}{\Delta}A_6Q_2[e^{\beta_2\pi}+e^{-\beta_2\pi}]\right\}\\
&\qquad\pm\frac{2\pi}{\Delta}A_1[\sqrt{-1}\beta_1+\dot{u}_1(0)][A_6Q_1+A_1^2Q_3]+\text{terms in }\varepsilon\\
&\qquad\text{and cubic and higher degree terms }A_1,\ A_3\text{ and }A_6,\\
0 &= \sqrt{-1}v_3(0)A_3\left\{e^{\beta_2\pi}-e^{-\beta_2\pi}+\frac{\pi}{\Delta}A_6Q_2[e^{\beta_2\pi}+e^{-\beta_2\pi}]\right\}\\
&\qquad+\frac{\sqrt{-1}\pi}{\Delta}A_6\left\{\pm2v_1(0)A_1Q_1+v_3(0)A_3Q_2[e^{\beta_2\pi}+e^{-\beta_2\pi}]\right\}\\
&\qquad+\text{terms in }\varepsilon\text{ and cubic and higher degree terms in }A_1,\\
&\qquad A_3\text{ and }A_6,\\
0 &= \pi[A\,A_6+A_1^2\phi_1(\pi)+A_3^2\phi_3(\pi)]+\text{terms in }\varepsilon\text{ and higher degree}\\
&\qquad\text{terms in }A_1,\ A_3\text{ and }A_6.
\end{aligned}\right\}\qquad(21)$$

Where the double sign occurs the $+$ is to be taken if β_1 is an even integer, and the $-$ if it is an odd integer.

The last equation of (21) can be solved for A_6 as a power series in A_1^2, A_3^2 and ε which vanishes with A_1 and A_6 but not with ε. We shall refer to this solution as (21c). When it is substituted in the second equation of (21) we obtain an equation in A_1, A_3 and ε which vanishes with A_1 and A_3 but not with ε, the terms of lowest degree being A_1^3 and A_3. As the coefficient of A_3 in this equation is a power series in a^2 with additional terms in $1/a^2$, it will be different from zero, in general, and the equation can be solved for A_3 as a power series in A_1^3 and ε which vanishes with A_1 but not with ε. Denote this series for A_3 by (21b). After substituting (21c) and (21b) in the first equation of (21), we obtain an equation (21a) in A_1 and ε which vanishes with these terms and in which the lowest power of A_1 alone is A_1^3. The coefficient of A_1^3 in this equation is a power series in a^2 with additional terms in $1/a^2$ and will, in general, be different from zero. Hence, (21a) can be solved for A_1 as a power series in $\varepsilon^{\frac{1}{3}}$ which vanishes with ε and converges for $|\varepsilon|$ sufficiently small. There are three solutions for A_1, but only one is real, the other two being complex. When this power series for A_1 is substituted in (21b) and (21c), A_3 and A_6 are likewise power series in $\varepsilon^{\frac{1}{3}}$. Therefore, when β_1 is an integer, periodic solutions exist having the form

$$x=\sum_{j=1}^{\infty} x^{(j)}\,\varepsilon^{\frac{j}{3}},\quad y=\sum_{j=1}^{\infty} y^{(j)}\,\varepsilon^{\frac{j}{3}},\quad w=\sum_{j=1}^{\infty} w^{(j)}\,\varepsilon^{\frac{j}{3}} \tag{22}$$

where each $x^{(j)}$, $y^{(j)}$ and $w^{(j)}$ is separately periodic with the period 2π in τ. These solutions converge for ε sufficiently small numerically. There is only one set of real solutions, the other two sets being complex.

In the practical construction of the solutions it can be shown that x and y are odd in $\varepsilon^{\frac{1}{3}}$, and that w is even in $\varepsilon^{\frac{1}{3}}$. This property of the solutions can be deduced directly from the differential equations (7) by an argument similar to that used at the end of § 4 if we replace ε in that section by $\varepsilon^{\frac{1}{3}}$. Then we may consider the solutions to have the form

$$x=\sum_{j=0}^{\infty} x^{(2j+1)}\,\varepsilon^{\frac{2j+1}{3}},\quad y=\sum_{j=0}^{\infty} y^{(2j+1)}\,\varepsilon^{\frac{2j+1}{3}},\quad w=\sum_{j=1}^{\infty} w^{(2j)}\,\varepsilon^{\frac{2j}{3}}. \tag{23}$$

§ 6. *Proof that all the Periodic Orbits are Symmetrical.*

We shall now consider the existence of periodic orbits in which the infinitesimal body is projected from the xy-plane in a direction which may not be perpendicular to the x-axis, and from a point which may not lie on the x-axis. These orbits, if they exist, are called the general orbits. They have the initial values

$$x(0)=\alpha_1,\ \dot{x}(0)=\alpha_2,\ y(0)=\alpha_3,\ y(0)=\alpha_4,\ w(0)=0,\ \dot{w}(0)=\alpha_5. \tag{24}$$

The initial value of w can be taken to be zero without loss of geometric generality since the infinitesimal body must cross the xy-plane if the motion is to be periodic, and hence the initial time can be chosen as the time when the infinitesimal body crosses the xy-plane. Sufficient conditions that solutions having the initial values (24) shall be periodic are

$$\left.\begin{array}{lll} x(2\pi)-x(0)=0, & y(2\pi)-y(0)=0, & w(2\pi)-w(0)=0, \\ \dot{x}(2\pi)-\dot{x}(0)=0, & \dot{y}(2\pi)-\dot{y}(0)=0, & \dot{w}(2\pi)-\dot{w}(0)=0. \end{array}\right\} \tag{25}$$

We shall now show that one of these conditions, viz., $\dot{w}(2\pi)-\dot{w}(0)=0$, can be suppressed.

So far no use has been made of the integral (5). It is by means of this integral that we show that one of the conditions in (25) is redundant. When (5) is transformed by the substitutions (6), it takes the form

$$\dot{x}^2+\dot{y}^2+(\dot{\psi}+\dot{w})^2+F(x,\, y^2,\, w;\ \varepsilon)=0, \tag{26}$$

where F is a power series in x, y^2, w and ε having periodic coefficients. Let us make in (26) the usual substitutions

$$\left.\begin{array}{lll} x=x(0)+\bar{x}, & y=y(0)+\bar{y}, & w=0+\bar{w}, \\ \dot{x}=\dot{x}(0)+\dot{\bar{x}}, & y=\dot{y}(0)+\dot{\bar{y}}, & \dot{w}=\dot{w}(0)+\dot{\bar{w}}, \end{array}\right\} \tag{27}$$

where $\bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}, \bar{w}$ and $\dot{\bar{w}}$ vanish at $\tau=0$. We shall denote the equation resulting from substituting (27) in (26) by (26a). By putting $\tau=0$ we obtain from (26a) an equation (26b) connecting the terms of (26a) which are independent of $\bar{x}, \ldots, \dot{\bar{w}}$. On substituting (26b) in (26a) we obtain an equation

$$G(\bar{x},\, \dot{\bar{x}},\, \bar{y},\, \dot{\bar{y}},\, \bar{w},\, \dot{\bar{w}})=0, \tag{28}$$

in which at $\tau=0$ or 2π there are no terms independent of the arguments indicated. The coefficient of $\dot{\bar{w}}(2\pi)$ in this equation is $2[a+\dot{w}(0)]$, and it vanishes only when $\dot{w}(0)=-a$. If $w(0)=-a$ then $\dot{z}(0)=0$, since $z=\psi+w$ and $\dot{\psi}(0)=a$, and as $z(0)=0$ it follows that $z\equiv 0$. As we desire solutions which are not identically zero we must take $\dot{w}(0)\neq -a$. Therefore the coefficient of $\dot{\bar{w}}(2\pi)$ in (28) is not zero at $\tau=2\pi$, and by the theory of implicit functions this equation can be solved uniquely for $\dot{w}(2\pi)$ as a power series

$$\dot{\bar{w}}(2\pi)=H\{\bar{x}(2\pi),\ \dot{\bar{x}}(2\pi),\ \bar{y}(2\pi),\ \dot{\bar{y}}(2\pi),\ \bar{w}(2\pi)\}, \tag{29}$$

which vanishes with the arguments indicated and converges for sufficiently small values of the moduli of these arguments. When the first five conditions of (25) are satisfied it follows from (29) that the arguments of H are all zero, and therefore $\dot{\bar{w}}(2\pi)=0$. Then, since $\dot{\bar{w}}(2\pi)=\dot{w}(2\pi)-\dot{w}(0)$, it follows that the condition $\dot{w}(2\pi)-\dot{w}(0)=0$ is a consequence of the other conditions in (25) and may be suppressed.

We shall now integrate equations (7) as power series in α_i $(i=1,\ldots,5)$ and ε. The differential equations from which the linear terms are obtained are the same as the equations of variation, and consequently the solutions are the same as (9) and (11). When the initial conditions (24) are imposed on these solutions, we find that $A_5=0$ and that the remaining A_i are linear functions of α_i which vanish with α_i. Then, as in § 4, it is more convenient to integrate (7) as power series in A_i and ε. On imposing the necessary conditions of (25) upon (9) and (11) we obtain

$$\left.\begin{aligned}
0 &= A_1[e^{2\sqrt{-1}\beta_1\pi}-1]+A_2[e^{-2\sqrt{-1}\beta_1\pi}-1]+A_3[e^{2\beta_2\pi}-1]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +A_4[e^{-2\beta_2\pi}-1]+\ldots,\\
0 &= [\sqrt{-1}\,\beta_1+\dot{u}_1(0)]\left\{A_1[e^{2\sqrt{-1}\beta_1\pi}-1]-A_2[e^{-2\sqrt{-1}\beta_1\pi}-1]\right\}\\
&\qquad +[\beta_2 v_3(0)+\dot{v}_3(0)]\left\{A_3[e^{2\beta_2\pi}-1]-A_4[e^{-2\beta_2\pi}-1]\right\}+\ldots,\\
0 &= \sqrt{-1}\,v_1(0)\left\{A_1[e^{2\sqrt{-1}\beta_1\pi}-1]-A_2[e^{-2\sqrt{-1}\beta_1\pi}-1]\right\}\\
&\qquad\qquad +v_3(0)\left\{A_3[e^{2\beta_2\pi}-1]-A_4[e^{-2\beta_2\pi}-1]\right\}+\ldots,\\
0 &= [-\beta_1 v_1(0)+\sqrt{-1}\,\dot{v}_1(0)]\left\{A_1[e^{2\sqrt{-1}\beta_1\pi}-1]+A_2[e^{-2\sqrt{-1}\beta_1\pi}-1]\right\}\\
&\qquad +[\beta_2 v_2(0)+\dot{v}_3(0)]\left\{A_3[e^{2\beta_2\pi}-1]+A_4[e^{-2\beta_2\pi}-1]\right\}+\ldots,\\
0 &= 2\pi A\,A_6\,\dot{\phi}(0)+\ldots\ldots
\end{aligned}\right\}\quad(30)$$

The determinant of the coefficients of A_i is

$$2\pi\Delta A\,\dot{\phi}(0)\,[e^{2\sqrt{-1}\beta_1\pi}-1]\,[e^{-2\sqrt{-1}\beta_1\pi}-1]\,[e^{2\beta_2\pi}-1]\,[e^{-2\beta_2\pi}-1],\qquad(31)$$

and it vanishes only when β_1 is an integer since β_2 is real and Δ, A and $\dot{\phi}(0)$ are different from zero for a not zero but sufficiently small numerically. If β_1 is not an integer the determinant (31) is different from zero and equations (30) can be solved uniquely for A_i as power series in ε which vanish with ε and converge for $|\varepsilon|$ sufficiently small. Hence, when β_1 is not an integer, the general orbits exist uniquely and have the same form as the symmetrical orbits. Since the general orbits include the symmetrical and both classes of orbits are unique, then all the periodic orbits are symmetrical when β_1 is not an integer.

When β_1 is an integer the coefficients of A_1 and A_2 in the first four equations of (30) vanish and in order to discuss the existence of general orbits, it is necessary to consider the terms of higher degree in (30). In addition to the linear terms (9) we require, as in § 5, the quadratic terms in $A_i (i=1, 2, 3, 4, 6)$ which contain τ as a factor, and the terms of x containing τ as a factor and of the third degree in A_1 and A_2. These terms are found in the same way as the corresponding terms were determined in §5. They have the form

$$\left.\begin{aligned}
x_{11} &= \frac{1}{\Delta} A_6 Q_1 \tau\,[A_1 e^{\sqrt{-1}\,\beta_1\tau} u_1 - A_2 e^{-\sqrt{-1}\,\beta_1\tau} u_2] \\
&\qquad + \frac{\sqrt{-1}}{\Delta} A_6 Q_2 \tau\,[A_3 e^{\beta_2\tau} u_3 - A_4 e^{-\beta_2\tau} u_4], \\
y_{11} &= \frac{\sqrt{-1}}{\Delta} A_6 Q_1 \tau\,[A_1 e^{\sqrt{-1}\,\beta_1\tau} v_1 + A_2 e^{-\sqrt{-1}\,\beta_1\tau} v_2] \\
&\qquad + \frac{\sqrt{-1}}{\Delta} A_6 Q_2 \tau\,[A_3 e^{\beta_2\tau} v_3 + A_4 e^{-\beta_2\tau} v_4], \\
w_{11} &= \tau\,[A_1 A_2 \phi_1 + A_3 A_4 \phi_3] + A_6^2\,[\text{non-periodic terms}] \\
&\qquad + \text{periodic terms}, \\
x_{30} &= \frac{1}{\Delta} A_1 A_2 Q_3 \tau\,[A_1 e^{\sqrt{-1}\,\beta_1\tau} u_1 - A_2 e^{\sqrt{-1}\,\beta_1\tau} u_2].
\end{aligned}\right\} \quad (32)$$

When the necessary periodicity conditions of (25) are imposed on equations (32), (9) and (11), we obtain

$$\left.\begin{aligned}
0 &= A_3[e^{2\beta_2\pi} - 1] + A_4[e^{-2\beta_2\pi} - 1] + \frac{2\pi}{\Delta} A_6 Q_1 [A_1 - A_2] \\
&\quad + \frac{2\sqrt{-1}\,\pi}{\Delta} A_6 Q_2 [A_3 e^{2\beta_2\pi} - A_4 e^{-2\beta_2\pi}] + \frac{2\pi}{\Delta} A_1 A_2 Q_3 [A_1 - A_2] \\
&\quad + \text{terms in } \varepsilon \text{ and higher degree terms in } A_i, \\
0 &= [\beta_2 + \dot{u}_3(0)]\left\{A_3[e^{2\beta_2\pi} - 1] - A_4[e^{-2\beta_2\pi} - 1]\right\} \\
&\quad + \frac{2\pi}{\Delta}[\sqrt{-1}\,\beta_1 + \dot{u}_1(0)]\,[A_6 Q_1 + A_1 A_2 Q_3]\,[A_1 + A_2] \\
&\quad + \frac{2\sqrt{-1}\,\pi}{\Delta} A_6 Q_2 [\beta_2 + \dot{u}_3(0)]\,[A_3 e^{2\beta_2\pi} + A_4 e^{-2\beta_2\pi}] \\
&\quad + \text{terms in } \varepsilon \text{ and higher degree terms in } A_i, \\
0 &= v_3(0)\left\{A_3[e^{2\beta_2\pi} - 1] - A_4[e^{-2\beta_2\pi} - 1]\right\} \\
&\quad + \frac{2\sqrt{-1}\,\pi}{\Delta} v_1(0)\, A_6 Q_1 [A_1 + A_2] \\
&\quad + \frac{2\sqrt{-1}\,\pi}{\Delta} v_3(0)\, A_6 Q_2 [A_3 e^{2\beta_2\pi} + A_4 e^{-2\beta_2\pi}] \\
&\quad + \text{terms in } \varepsilon \text{ and higher degree terms in } A_i, \\
0 &= [\beta_2 v_3(0) + \dot{v}_3(0)]\left\{A_3[e^{2\beta_2\pi} - 1] + A_4[e^{-2\beta_2\pi} - 1]\right\} \\
&\quad + \frac{2\sqrt{-1}\,\pi}{\Delta} A_6 Q_1 [\sqrt{-1}\,\beta_1 v_1(0) + \dot{v}_1(0)]\,[A_1 - A_2] \\
&\quad + \frac{2\sqrt{-1}\,\pi}{\Delta} A_6 Q_2 [\beta_2 v_3(0) + \dot{v}_3(0)]\,[A_3 e^{2\beta_2\pi} - A_4 e^{-2\beta_2\pi}] \\
&\quad + \text{terms in } \varepsilon \text{ and higher degree terms in } A_i, \\
0 &= 2\pi A A_6 \dot{\phi}(0) + 2\pi\,[A_1 A_2 \phi_1(0) + A_3 A_4 \phi_3(0)] \\
&\quad + \text{terms in } \varepsilon \text{ and higher degree terms in } A_i.
\end{aligned}\right\} \quad (33)$$

The last equation of (33) can be solved for A_6 as a power series in A_1, A_2, A_3, A_4 and ε, and the terms of lowest degree in A_i are $A_1 A_2$ and $A_3 A_4$. The determinant of the coefficients of A_3 and A_4 in the third and fourth equations is

$$2 v_3(0)\,[\beta_2 v_3(0)+\dot{v}_3(0)]\,[e^{2\beta_2\pi}-1]\,[e^{-2\beta_2\pi}-1],$$

and it is different from zero for a^2 sufficiently small since β_2 is real. Hence these two equations, when A_6 has been eliminated by the last equation, can be solved for A_3 and A_4 as power series in A_1, A_2 and ε in which the terms of lowest degree in A_1 and A_2 are $A_1^2 A_2$ and $A_1 A_2^2$. When these solutions for A_3, A_4 and A_6 are substituted in the second equation, the terms independent of ε contain $A_1^2 A_2$ or $A_1 A_2^2$ as a factor. Hence this equation can be solved for A_2 as a power series in A_1 and ε, and the term of lowest degree in A_1 is linear in A_1. Finally, when the solutions for A_2, A_3, A_4 A_6 are substituted in the first equation, we obtain an equation in A_1 and ε in which the terms of lowest degree in A_1 is A_1^3. As in (25a) the coefficient of A_1^3 in this equation is a power series in a^2 with additional terms in $1/a^2$ and will, in general, be different from zero; therefore this equation can be solved for A_1^3 as a power series in $\varepsilon^{\frac{1}{3}}$ which vanishes with ε and converges for $|\varepsilon|$ sufficiently small. There are three solutions as in the symmetrical orbits, but only one solution is real, the other two being complex. Since the real solutions for the general orbits and the symmetrical orbits are both unique and since the general include the symmetrical orbits, all the periodic orbits are symmetrical when β_1 is an integer. Consequently all the periodic orbits are symmetrical whether β_1 is an integer or not.

§ 7. *Constructions of the Solutions when β_1 is not an Integer.*

Let us substitute (18) in (7) and equate the coefficients of the various powers of ε. We obtain a series of differential equations which can be integrated step by step and the constants of integration arising at each step can be determined, as we shall show, so that the solutions shall be periodic and shall satisfy the symmetrical initial conditions

$$x=y=w=0 \text{ at } \tau=0.$$

When these initial conditions are imposed on the solutions (18) we obtain

$$\left.\begin{aligned} \dot{x}_{2j+1}(0)=y_{2j+1}(0)=0, &\qquad (j=0,\ \ldots.\infty), \\ w_{2j}(0)=0, &\qquad (j=1,\ \ldots.\infty). \end{aligned}\right\} \tag{34}$$

The differential equations for the terms in ε are

$$\left.\begin{aligned} x_1-2\sqrt{\tfrac{1}{8}(1+\delta)}\,\dot{y}_1-[\tfrac{1}{8}(1+\delta)+X_0^{(0)}]\,x_1&=X_0^{(1)}, \\ \ddot{y}_1+2\sqrt{\tfrac{1}{8}(1+\delta)}\,\dot{x}_1-[\tfrac{1}{8}(1+\delta)+Y_0^{(0)}]\,y_1&=0, \end{aligned}\right\}$$

where $X_0^{(1)}$ is the same even power series as that represented in (7) by the same notation. The complementary functions of (35) are the same as (9), and the particular integrals can be obtained by the method of the variation of parameters. The complete solutions are thus found to be

$$\left.\begin{aligned} x_1 &= A_1^{(1)} e^{\sqrt{-1}\beta_1\tau} u_1 + A_2^{(1)} e^{-\sqrt{-1}\beta_1\tau} u_2 + A_3^{(1)} e^{\beta_2\tau} u_3 + A_4^{(1)} e^{-\beta_2\tau} u_4 + C_1(\tau), \\ y_1 &= \sqrt{-1}\,[A_1^{(1)} e^{\sqrt{-1}\beta_1\tau} v_1 - A_2^{(1)} e^{-\sqrt{-1}\beta_1\tau} v_2] + A_3^{(1)} e^{\beta_2\tau} v_3 - A_4^{(1)} e^{-\beta_2\tau} v_4 + S_1(\tau), \end{aligned}\right\} \quad (35)$$

where $A_i^{(1)}$ $(i=1, 2, 3, 4)$ are the constants of integration, and $C_1(\tau)$ and $S_1(\tau)$ are triply even and triply odd series, respectively. As such series occur frequently in the construction, we shall denote them by $C_j(\tau)$ and $S_j(\tau)$, respectively. Since β_1 is not an integer in this case and β_2 is real, the terms of the complementary function must be suppressed in (35) by choosing $A_i^{(1)}=0$, $(i=1, 2, 3, 4)$, and therefore the desired solutions become

$$x_1 = C_1(\tau), \quad y_1 = S_1(\tau). \qquad (36)$$

The differential equation for the terms in ε^2 is

$$\ddot{w}_2 + W_0 w_2 = W_2,$$

where W_2 is a triply odd series. The complete solution of this equation is obtained by employing the method of the variation of parameters, and it is found to be

$$w_2 = B_1^{(2)}\phi + B_2^{(2)}[\chi + A\tau\phi] - ap_2\tau\phi + S_2(\tau), \qquad (37)$$

where $B_1^{(2)}$ and $B_2^{(2)}$ are constants of integration, and p_2 is a power series in a^2 with real constant coefficients. In order to satisfy the periodicity and the initial conditions (34), the constants of integration must have the values

$$B_1^{(2)} = 0, \quad B_2^{(2)} = \frac{ap_2}{A}, \qquad (38)$$

in which case the solution (37) becomes

$$w_2 = \frac{1}{a^2}\bar{S}_2(\tau), \qquad (39)$$

where $\bar{S}_2(\tau)$ is a triply odd power series.

The remaining steps of the integration can be carried on in the same way. The solutions for x_{2j-1} and y_{2j-1} are obtained from the particular integrals alone since β_2 is real and β_1 is assumed to be a number which is not an integer. These particular integrals are triply even and triply odd series, respectively, except for a factor $a^{-2(j-2)}$ which is introduced through the factor a^{-2} in (39). The general solution for the w_{2j} is similar to (37) and it can be made periodic

by a proper choice of the constants of integration as in (38). At the steps $2j-1$ and $2j$ the solutions are

$$\left.\begin{aligned} x_{2j-1} &= \frac{1}{a^{2j-4}} C_{2j-1}(\tau), \\ y_{2j-1} &= \frac{1}{a^{2j-4}} S_{2j-1}(\tau), \\ w_{2j} &= \frac{1}{a^{2j}} S_{2j}(\tau). \end{aligned}\right\} \tag{40}$$

§ 8. *Construction of the Solutions when β_1 is an Integer.*

Let us substitute equations (23) in (7) and equate the coefficients of the various powers of $\varepsilon^{\frac{1}{2}}$. As in the previous section we obtain a series of differential equations which can be integrated step by step, and the constants of integration can be determined, as we shall show, so that the solutions shall be periodic and shall satisfy the symmetrical initial conditions

$$\dot{x}(0) = y(0) = w(0) = 0.$$

When these conditions are imposed upon (23) we obtain

$$\left.\begin{aligned} \dot{x}^{(2j+1)}(0) = y^{(2j+1)}(0) &= 0, \qquad (j=0, \ldots . \infty), \\ w^{(2j)}(0) &= 0, \qquad (j=1, \ldots . \infty). \end{aligned}\right\} \tag{41}$$

The differential equations for the terms in $\varepsilon^{\frac{1}{2}}$ are the same as the first two equations of (8) if we use the superscript 1 on x and y. Since β_1 is assumed to be an integer, the solutions of these equations which are periodic and which satisfy (41) are

$$\left.\begin{aligned} x^{(1)} &= A_1^{(1)} [e^{\sqrt{-1}\beta_1\tau} u_1 + e^{-\sqrt{-1}\beta_1\tau} u_2], \\ y^{(1)} &= \sqrt{-1} A_1^{(1)} [e^{\sqrt{-1}\beta_1\tau} v_1 - e^{-\sqrt{-1}\beta_1\tau} v_2], \end{aligned}\right\} \tag{42}$$

where $A_1^{(1)}$ is a constant which is undetermined at this step. The solutions (42) are real if $A_1^{(1)}$ is real.

The differential equation for the terms in $\varepsilon^{\frac{2}{2}}$ is

$$\ddot{w}^{(2)} + W_0 w^{(2)} = W^{(2)} = (A_1^{(1)})^2 S^{(2)}(\tau). \tag{43}$$

As functions similar to $S^{(2)}(\tau)$ occur frequently in the construction of the remaining solutions, we shall define the function $S^{2i}(\tau)$ to be a homogeneous polynomial of degree $2i$ in the exponentials $e^{\sqrt{-1}\beta_1\tau}$ and $e^{-\sqrt{-1}\beta_1\tau}$ of which the coefficients are power series in odd powers of a with sums of sines and $\sqrt{-1}$ times cosines of odd multiples of τ in the coefficients. The highest multiple of τ in the coefficient of a^{2k+1} is $2k+1$. The coefficients of $[e^{\sqrt{-1}\beta_1\tau}]^j\,[e^{-\sqrt{-1}\beta_1\tau}]^k$ and

$[e^{\sqrt{-1}\beta_1\tau}]^k[e^{-\sqrt{-1}\beta_1\tau}]^j$, $j+k=2i$, differ only in the sign of $\sqrt{-1}$. The part of $S^{(2i)}(\tau)$ which is independent of the exponentials is a triply odd power series. If the exponentials are expressed in trigonometric form, then

$$S^{(2i)}(\tau)=\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{h=0}^{i} a^{2j+1}S_{h,j,l}^{(2i)} \sin \{2h\beta_1\pm(2l+1)\}\tau,$$

where $S_{h,j,l}^{(2i)}$ are real constants: that is, $S^{(2i)}(\tau)$ is the same as a triply odd power series except that the highest multiple of τ in the coefficient of a^{2j+1} is not $2j+1$ but $2(j+i\beta_1)+1$.

The general solution of (43) is found in precisely the same way as (37) was obtained. It has the form

$$w^{(2)}=B_1^{(2)}\phi+B_2^{(2)}[\chi+A\tau\phi]-a(A_1^{(1)})^2p^{(2)}\tau\phi+(A_1^{(1)})^2S_0^{(2)}(\tau), \tag{44}$$

where $p^{(2)}$ is a power series in a^2 with real constant coefficients and $S_0^{(2)}(\tau)$ is similar to $S^{(2)}(\tau)$. On imposing the periodicity and the initial conditions (41), we have

$$B_1^{(2)}=0, \quad B_2^{(2)}=\frac{a}{A}(A_1^{(1)})^2p^{(2)}. \tag{45}$$

Then the desired solution of (43) takes the form

$$w^{(2)}=\frac{1}{a^2}(A_1^{(1)})^2\bar{S}^{(2)}(\tau),$$

where $\bar{S}^{(2)}(\tau)$ is similar to $S^{(2)}(\tau)$. Thus $A_1^{(1)}$ remains undetermined at this step also, and the terms in ϵ^3 must be considered.

The left members of the differential equations for the terms in ϵ^3 are the same as the first two equations in (8) if we use the superscript 3 on x and y. Let us denote the right members by $X^{(3)}$ and $Y^{(3)}$, respectively. Then

$$X^{(3)}=(A_1^{(1)})^3\rho^{(3)}+X_0^{(1)}, \quad Y^{(3)}=\sqrt{-1}(A_1^{(1)})^3\sigma^{(3)}. \tag{46}$$

The function $X_0^{(1)}$ is the same triply even power series as that denoted in (7) by the same notation. The functions $\rho^{(2i+1)}$ and $\sigma^{(2i+1)}$ are homogeneous polynomials of degree $2i+1$ in the exponentials $e^{\sqrt{-1}\beta_1\tau}$ and $e^{-\sqrt{-1}\beta_1\tau}$, and the coefficients are power series similar in form to u_1 and u_2 in (9). In $\rho^{(2i+1)}$ the coefficients of $[e^{\sqrt{-1}\beta_1\tau}]^j[e^{-\sqrt{-1}\beta_1\tau}]^k$ and $[e^{\sqrt{-1}\beta_1\tau}]^k[e^{-\sqrt{-1}\beta_1\tau}]^j$, $j+k=2i+1$, differ only in the sign of $\sqrt{-1}$; while in $\sigma^{(2i+1)}$ the coefficients of $[e^{\sqrt{-1}\beta_1\tau}]^j[-e^{-\sqrt{-1}\beta_1\tau}]^k$ and $[e^{\sqrt{-1}\beta_1\tau}]^k[-e^{-\sqrt{-1}\beta_1\tau}]^j$ differ only in the sign of $\sqrt{-1}$.

The complementary functions of the differential equations in $x^{(3)}$ and $y^{(3)}$ are the same as (9). Since $X^{(3)}$ and $Y^{(3)}$ contain terms which have exactly the same period as the periodic parts of the complementary functions, the par-

ticular integrals will contain non-periodic terms. By the method of the variation of parameters we find that the general solutions for $x^{(3)}$ and $y^{(3)}$ are

$$\left.\begin{aligned} x^{(3)} &= A_1^{(3)} e^{\sqrt{-1}\beta_1\tau} u_1 + A_2^{(3)} e^{-\sqrt{-1}\beta_1\tau} u_2 + A_3^{(3)} e^{\beta_2\tau} u_3 + A_4^{(3)} e^{-\beta_2\tau} u_4 \\ &\qquad + \tau[(A_1^{(1)})^3 P_1^{(3)} + P_2^{(3)}][e^{\sqrt{-1}\beta_1\tau} u_1 - e^{-\sqrt{-1}\beta_1\tau} u_2] + \bar{\rho}^{(3)}, \\ y^{(3)} &= \sqrt{-1}[A_1^{(3)} e^{\sqrt{-1}\beta_1\tau} v_1 - A_2^{(3)} e^{-\sqrt{-1}\beta_1\tau} v_2] + A_3^{(3)} e^{\beta_2\tau} v_3 - A_4^{(3)} e^{-\beta_2\tau} v_4 \\ &\qquad + \sqrt{-1}\tau[(A_1^{(1)})^3 P_1^{(3)} + P_2^{(3)}][e^{\sqrt{-1}\beta_1\tau} v_1 + e^{-\sqrt{-1}\beta_1\tau} v_2] + \sqrt{-1}\bar{\sigma}^{(3)}, \end{aligned}\right\} \quad (47)$$

where $A_i^{(3)}$ $(i=1, 2, 3, 4)$ are the constants of integration, $P_i^{(3)}$ $(i=1, 2)$ are power series in a^2 with constant coefficients, and $\bar{\rho}^{(3)}$ and $\bar{\sigma}^{(3)}$ are similar to $\rho^{(3)}$ and $\sigma^{(3)}$, respectively. In order that the solutions shall be periodic, then

$$A_3^{(3)} = A_4^{(3)} = 0, \quad P_1^{(3)}(A_1^{(1)})^3 + P_2^{(3)} = 0.$$

The solutions for $A_1^{(1)}$ are

$$A_1^{(1)} = P_1^{(1)}, \quad \omega P_1^{(1)} \text{ or } \omega^2 P_1^{(1)},$$

where $P_1^{(1)}$ is a power series in a^2 with real constant coefficients, and ω, ω^2 are the imaginary cube roots of unity. As the imaginary solutions for $A_1^{(1)}$ lead to imaginary orbits, we retain only the real solution. Then, on imposing the initial conditions (41) on (47) we find that $A_1^{(3)} = A_2^{(3)}$, and the desired solutions for $x^{(3)}$ and $y^{(3)}$ become

$$\begin{aligned} x^{(3)} &= A_1^{(3)}[e^{\sqrt{-1}\beta_1\tau} u_1 + e^{-\sqrt{-1}\beta_1\tau} u_2] + \bar{\rho}^{(3)}, \\ y^{(3)} &= \sqrt{-1} A_1^{(3)}[e^{\sqrt{-1}\beta_1\tau} v_1 - e^{-\sqrt{-1}\beta_1\tau} v_2] + \sqrt{-1}\bar{\sigma}^{(3)}. \end{aligned}$$

The constant $A_1^{(3)}$ remains arbitrary at this step and the terms up to $\varepsilon^{\frac{5}{3}}$ must be considered before it can be determined. The solutions for $x^{(3)}$ and $y^{(3)}$ are real provided $A_1^{(3)}$ is real, which is found to be the case.

The general solutions of the differential equation in $\varepsilon^{\frac{4}{3}}$ is obtained in the same way as (44) was found, and it can be made to satisfy the periodicity and initial conditions by a choice of the constants of integration as in (45). When these conditions have been imposed, the solution for $w^{(4)}$ takes the form

$$w^{(4)} = \frac{1}{a^4}[A_1^{(3)} S^{(2)} + S^{(4)}].$$

The differential equations in $\varepsilon^{\frac{5}{3}}$ are the same as (8) in the left members if we use the superscripts 5. If we denote the right members by $X^{(5)}$ and $Y^{(5)}$, respectively, then

$$X^{(5)} = \frac{1}{a^2}[A_1^{(3)} \rho_5^{(3)} + \rho_5^{(5)}], \quad Y^{(5)} = \frac{\sqrt{-1}}{a^2}[A_1^{(3)} \sigma_5^{(3)} + \sigma_5^{(5)}],$$

where $\rho_5^{(3)}$, $\rho_5^{(5)}$ and $\sigma_5^{(3)}$, $\sigma_5^{(5)}$ are similar to $\rho^{(2k+1)}$ and $\sigma^{(2k+1)}$, respectively. The

complete solutions of the equations in $x^{(5)}$ and $y^{(5)}$ are readily found in the same way as (47) were obtained. They are:

$$\left.\begin{aligned} x^{(5)} &= A_1^{(5)} e^{\sqrt{-1}\beta_1\tau} u_1 + A_2^{(5)} e^{-\sqrt{-1}\beta_1\tau} u_2 + A_3^{(5)} e^{\beta_2\tau} u_3 + A_4^{(5)} e^{-\beta_2\tau} u_4 \\ &\quad + \frac{\tau}{a^2} [P_1^{(5)} A_1^{(3)} + P_2^{(5)}] [e^{\sqrt{-1}\beta_1\tau} u_1 - e^{-\sqrt{-1}\beta_1\tau} u_2] + \frac{1}{a^2} \rho^{(5)}, \\ y^{(5)} &= \sqrt{-1} [A_1^{(5)} e^{\sqrt{-1}\beta_1\tau} v_1 - A_2^{(5)} e^{-\sqrt{-1}\beta_1\tau} v_2] + A_3^{(5)} e^{\beta_2\tau} v_3 - A_4^{(5)} e^{-\beta_2\tau} v_4 \\ &\quad + \frac{\sqrt{-1}\tau}{a^2} [P_1^{(5)} A_1^{(3)} + P_2^{(5)}] [e^{\sqrt{-1}\beta_1\tau} v_1 + e^{-\sqrt{-1}\beta_1\tau} v_2] + \frac{\sqrt{-1}}{a^2} \sigma^{(5)}, \end{aligned}\right\} \quad (48)$$

where $P_i^{(5)}$ $(i=1, 2)$ are power series in a^2 with real constant coefficients. The solutions for $x^{(5)}$ and $y^{(5)}$ can be made periodic by putting $A_3^{(5)} = A_4^{(5)} = 0$, and then choosing $A_1^{(3)}$ so that

$$P_1^{(5)} A_1^{(3)} + P_2^{(5)} = 0,$$

from which it follows that $A_1^{(3)}$ is real. From the initial conditions (41) we have $A_1^{(5)} = A_2^{(5)}$.

The remaining steps of the integration can be carried on in the same way. The solution at the step $2j$ can be made to satisfy the periodicity and the initial conditions by a proper choice of the constants of integration arising at that step. The solution for w_{2j} contains the constant $A_1^{(2j-1)}$ which is not determined until the step $2j+1$. The non-periodic parts of the complete solutions at the step $2j+1$ are similar to those in (48), and $A_1^{(2j-1)}$ enters these solutions with the same coefficient as $A_1^{(3)}$ enters (48) except for the factor $1/a^{2j-2}$. The solutions can be made to satisfy the initial and periodicity conditions by putting

$$A_1^{(2j+1)} = A_2^{(2j+1)}, \quad A_3^{(2j+1)} = A_4^{(2j+1)} = 0,$$

and then determine the $A_1^{(2j-1)}$ so that the coefficient of τ shall be zero. The desired solutions at the steps $2j$ and $2j+1$ are

$$\begin{aligned} w^{(2j)} &= \frac{1}{a^{2j}} S^{2j}(\tau), \\ x^{(2j+1)} &= A_1^{(2j+1)} [e^{\sqrt{-1}\beta_1\tau} u_1 + e^{-\sqrt{-1}\beta_1\tau} u_2] + \frac{1}{a^{2j-2}} \rho^{(2j+1)}, \\ y^{(2j+1)} &= \sqrt{-1} A_1^{(2j+1)} [e^{\sqrt{-1}\beta_1\tau} v_1 - e^{-\sqrt{-1}\beta_1\tau} v_2] + \frac{\sqrt{-1}}{a^{2j-2}} \sigma^{(2j+1)}. \end{aligned}$$

Thus the integration can be carried on to any desired degree of accuracy.

The solutions of (7), whether as power series in ε or $\varepsilon^{\frac{1}{3}}$, must satisfy the integral (5) identically in ε or $\varepsilon^{\frac{1}{3}}$. Thus this integral, besides being of use in proving that all the periodic orbits are symmetrical, serves as a check on the computations.

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